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Anti-intuitionism and paraconsistency

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Abstract

This paper aims to help to elucidate some questions on the duality between the intuitionistic and the paraconsistent paradigms of thought, proposing some new classes of anti-intuitionistic propositional logics and investigating their relationships with the original intuitionistic logics. It is shown here that anti-intuitionistic logics are paraconsistent, and in particular we develop a first anti-intuitionistic hierarchy starting with Johansson's dual calculus and ending up with Gödel's three-valued dual calculus, showing that no calculus of this hierarchy allows the introduction of an internal implication symbol. Comparing these anti-intuitionistic logics with well-known paraconsistent calculi, we prove that they do not coincide with any of these. On the other hand, by dualizing the hierarchy of the paracomplete (or maximal weakly intuitionistic) many-valued logics $(I^n)_{n\in\omega}$ we show that the anti-intuitionistic hierarchy $(I^{n*})_{n\in\omega}$ obtained from $(I^n)_{n\in\omega}$ *does* coincide with the hierarchy of the many-valued paraconsistent logics $(P^n)_{n\in\omega}$. Fundamental properties of our method are investigated, and we also discuss some questions on the duality between the intuitionistic and the paraconsistent paradigms, including the problem of self-duality. We argue that questions of duality quite naturally require refutative systems (which we call *elenctic* systems) as well as the usual demonstrative systems (which we call *deictic* systems), and multiple-conclusion logics are used as an appropriate environment to deal with them.

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1. Introduction

The question of the purported duality between intuitionistic and paraconsistent ways of thinking arises from time to time, supported by the view that intuitionistic logics are "false by default" (in the sense that a proposition and its negation can both be taken to be false), while paraconsistent logics are "true by default" (in the sense that a proposition and its negation can both be taken to be true). The very notion of duality involved in the discussion is far from clear (not to mention the general concepts of paraconsistent and intuitionistic paradigms), and thus the question could hardly be considered as solvable. This paper contributes to this problem by introducing anti-intuitionistic systems in a broad sense through a general dualization procedure. We focus first on the very notion of dualization, and then investigate several properties of the logics obtained, establishing a relationship between the original and dual logics and comparing the dual (anti-intuitionistic) logics with some well-known paraconsistent systems.

The concept of anti-intuitionism, proposed through the concept of a dual intuitionistic logic, was already mentioned in the forties (without giving any formalization) by Popper, cf. [26]. More or less by the same time paraconsistency was being engendered with the work of S. Jaśkowski and N. da Costa. The driving force behind this spontaneous common interest seems to be the surmise that there should be a logic for unrestricted reasoning from hypothesis (or for falsifying propositions, in the case of Popper), and in the search for evaluating hypothesis (or for falsification, in the case of Popper) it makes good sense to retain some propositions and their negations as true (or not falsified). This is not disconnected from the philosophical program of falsificationism of Popper (cf. [27]).

Although Popper disregarded such a logic as "too weak as to be useless" (cf. [25]), Miller in [22,23] defends that paraconsistent logic, or dual intuitionistic logic, indeed constitutes a suitable logic mate for falsificationism.

We are here considering the intuitionistic paradigm in a wide sense, embodying not only Heyting's system H but also intermediate logics (between H and classical logic C), Johansson's Calculus (which can be regarded as intuitionistic and paraconsistent[3]) and paracomplete or weakly-intuitionistic logics (cf. [33]).

The paraconsistent paradigm, on the other hand, is apparently more comprehensive and we can define a paraconsistent logic, in general, as any logic system L endowed with a symbol for negation for which the principle of Pseudo Scotus does not hold, i.e., for which there exist a theory Γ and formulas φ and ψ such that, $\Gamma, \varphi, \neg\varphi \nvdash_L \psi$ (for a thorough discussion see [10,11]); this distinction is partly due to the very distinct philosophical motivations of both schools, as history of logic teaches us.

The literature offers only very few notes about anti-intuitionism besides the mentioned ones; in [19] the author develops a *sequent calculus* (in the sense of Gentzen) for Heyting's dual calculus. There, a sequent is an expression $\varphi \Rightarrow \Gamma$, where the left side consists of a single formula only and the set of formulas Γ on the right side is interpreted disjunctively in the sense of multiple-conclusion logics (see [34]). The reason for giving a sequent calculus lies in the fact that Heyting's dual calculus has no internal implication symbol

[3] Although Johansson's logic is classified in [9] as *partially explosive*, and is not generally considered to be a paraconsistent logic, it is convenient for our purposes to consider it as intuitionistic and paraconsistent.

in the sense of [1]. In that paper the notion of Brouwerian algebras—the duals of Heyting algebras—are used to obtain sound and correct topological semantics for this sequent calculus. A study on the duality between paraconsistency and intuitionism is developed in [28] in the direction of [30], pointing to algebraic and categorial characterizations; we have however distinct interests and a different approach.

In [37] Heyting's dual calculus is also represented by means of a sequent calculus in Gentzen's sense. In addition to the results of [19], the author develops various proposals for duals of the intuitionistic Gentzen calculus *LJ*: *LDJ* is obtained by adding to *LJ* a (defined) implication symbol $\supset$; LDJ^- is obtained from *LDJ* by adding an operator of pseudo-difference here denoted by $-$ (reading $\varphi - \psi$ as "φ excludes ψ", as suggested in [37]) and $LDJ^-_{\not\supset}$ is the fragment of LDJ^- free of the connective $\supset$.

Besides proving other results as cut elimination and decidability for the various systems treated there, the paper best contribution to the question of duality is the discussion and comparison of these various systems as candidates of duals of *LJ*. While *LDJ* and LDJ^- could be seen as duals of *LJ* in the proof-theoretic sense of being singular in the antecedent of sequent rules rather than in the consequent (as *LJ* is), *LDJ* owns the Glivenko's property of sharing propositional theorems (but not counter-theorems) with classical logic. So the question of deciding "who is the real dual" is not immediately obvious. Although [24] claims that if a sentence φ is a theorem in Heyting's calculus H (i.e., deducible), then the dual sentence φ^* is a counter-theorem (i.e., refutable) in the dual calculus H^* (and even if this kind of property will indeed be stressed as the main characteristic of dualization) neither naive proof-theoretical symmetry (in which sense?) nor sharing properties with respect to classical reasoning (which properties?) will do the job of explaining duality.

It is necessary to guarantee from the beginning a precise notion of duality, which can only be given by defining appropriate translations and their effects on consequence relations. This is what [37] implicitly does, and concludes that the exact dual of *LJ* is $LDJ^-_{\not\supset}$, both being related by means of a translation.

This is the way we work here, generalizing at the same time the ideas of [37] and that of [30], towards a general method for dualizing logics. Based on this framework we then develop two basic hierarchies of anti-intuitionistic logics: the first is the hierarchy *AC* of the *anti-constructive* logics. This denomination can be understood taking into account that, as far as the intuitionistic philosophic program is committed to constructing truthood, our anti-constructive logics can be seen as committed to eliminating falsehood.[4] The second is the hierarchy *AP* of the *anti-paracomplete* logics. Although both advocate the suppression of the law of excluded middle, the philosophical program of intuitionism is more demanding than paracompleteness, for example to what concerns rejection of the law of double negation.

The hierarchy *AC* starts with Johansson's dual calculus, passes through Heyting's dual calculus, and ends up with the duals of Gödel's n-valued logics G^*_n.

Applying our procedure of dualization we obtain the dual of Johansson's calculus, denoted by J^*, which is (in a certain sense) intuitionistic and paraconsistent. This motivates

[4] Perhaps no logician or philosopher would describe the anti-constructive logics in more delicious terms than Sir Arthur Conan Doyle in "The Science of Deduction", the first chapter of his [13]: "Eliminate all other factors, and the one which remains must be the truth".

introducing the concept of *self-duality* (cf. Section 2). We show that Johansson's calculus and the hierarchy $I^n P^n$, $n \in \omega$ (cf. [16]) are not self-dual, while classical logic C, is indeed self-dual. Self-dual logics are, in a sense, totally symmetric with regard to truth-default and falsity-default semantical conditions (this terminology was introduced in [15], pp. 155–156). An interesting problem (which we only tackle partially here) is to characterize the class of self-dual logics.

In the same way we can define the dual calculus of Heyting's intuitionistic logic, denoted by H^*. This calculus is paraconsistent and satisfies *tertium non datur*. In the following we obtain Kripke semantics for these two calculi. It is quite clear for Heyting's dual H^* that such Kripke semantics is equivalent to algebraic semantics, where the values are elements of a Brouwerian algebra, see also [19,24,30].

To what concerns the dual-calculi of Gödel's hierarchy denoted by $(G_n^*)_{n\in\omega}$, see [15], we show that these are all paraconsistent and that, by interpreting the pseudo-difference $\varphi - \psi$ as $\varphi \wedge \neg\psi$, all classical tautologies are anti-intuitionistic tautologies and vice versa.

It was proven in earlier papers [19,37] that H^* do not allow for an internal implication symbol in the sense of Avron [1]. Considering the fact that a sentence does not imply its double negation, we can easily show that in $(G_n^*)_{n\in\omega}$ no internal implication symbol can be defined as well.

The second hierarchy *AP* consists, applying the same general procedure, of the duals of the maximal weakly intuitionistic logics (or paracomplete) $(I^n)_{n\in\omega}$, denoted by $(I^{n*})_{n\in\omega}$. We also obtain a hierarchy of anti-intuitionistic logics $(I^{n*})_{n\in\omega}$, which are paraconsistent. It will be shown that I^{n*} does not validate all classical tautologies, and that it is possible to introduce here an internal implication symbol $\to^*$ in the sense of [1].

Finally, we compare all these anti-intuitionistic logics with some well-known paraconsistent calculi, as with da Costa's hierarchy $(C_n)_{n\in\omega}$, and with $C_{\min}$, $C_{\lim}$, *Pac* and J_3 (cf. [9–11]). We will see that none of these paraconsistent logics coincide with the new systems introduced in our first hierarchy *AC*. On the other hand, it is promptly seen that our second hierarchy *AP* of anti-intuitionistic logics does coincide with the paraconsistent hierarchy $(P^n)_{n\in\omega}$ as mentioned in [33]. Based on the fact that the intuitionistic hierarchy $(I^n)_{n\in\omega}$ and the paraconsistent hierarchy $(P^n)_{n\in\omega}$ are axiomatizable, we can also establish, *via* dualization, counter-axiomatics (in a sense to be explained below) for each of these calculi.

Our results prove that, while anti-paracomplete logics are paraconsistent and coincide with known systems, anti-constructive logics constitute genuinely new paraconsistent logics.

This paper tackles one side of the question only, showing that the general method can be applied for several other calculi and studying in detail such classes of anti-intuitionistic logics.

2. An environment for dualization

As a first step for the notion of dualization of logical systems we state the propositional language L we will be interested and its dual language L^*.

Let L be a language with the logical connectives $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication) and (the 0-ary connective, or constant) $\bot$ (bottom). L^* denotes the dual language of L, formed by the connectives $\neg^*$ (dual-negation), $\vee$ (disjunction), $\wedge$ (conjunction), $-$ (pseudo-difference, or exclusion) and (the 0-ary connective, or constant) $\top$ (top).

The *dualizing translation*

$$*: L \to L^*$$

is then defined by induction on the complexity:

Definition 2.1.

(1) $\varphi^* := \varphi$, $\bot^* := \top$, for atomic φ,
(2) $(\neg\varphi)^* := \neg^*\varphi^*$, $(\varphi \wedge \psi)^* := \varphi^* \vee \psi^*$, $(\varphi \vee \psi)^* := \varphi^* \wedge \psi^*$, $(\varphi \to \psi)^* := \psi^* - \varphi^*$ for the non-atomic cases.

Recall that a negated formula in Heyting's intuitionistic logic is defined as $\neg\varphi := \varphi \to \bot$; therefore, in L^* the anti-intuitionistic negation is equivalent to $\neg^*\varphi := \top - \varphi$. It is natural, furthermore, to require the dualizing translation $*$ to be an involution, and in the obvious way we extend $*$ to the dual language L^*.

The L-formulas are build up in the usual way from the propositional variables in $\mathcal{P}$. We denote by $\mathit{Sent}(L)$ the set of all L-formulas (or sentences), and analogously by $\mathit{Sent}(L^*)$ the set of all L^*-formulas.

An environment for investigating dualization requires special attention to general consequence relations. Besides the usual concept of deductive systems, where theorems are produced in the usual way, the idea of dual logics may require a notion of refutation, where (contrary to the usual way) the so-called counter-theorems are excluded.

Although not customary in logic, traditional dialectic recognizes this general idea of refutation as a sophisticated form of reasoning, as in the Socratic methodological refutation known as *elenchos*, present in the early Socratic dialogues of Plato (see for example [4]). In contrast to what we can call *deictic systems* (that is, systems devoted to direct deductions, as usual) there are also the ones we could call *elenctic systems*,[5] which proceed by refuting conjectures. The notion of rejected proposition of a system had also been considered in connection to Aristotle's idea of relative rejection (demolishing) of certain propositions on the basis of other propositions by Jan Łukasiewicz, Jerzy Słupecki and their collaborators (see [35]). Their intention, however, was not of investigating duality, but directed towards obtaining decidability by means of showing that the non-theorems form a recursively enumerable set (a notion of rejection of this sort was also proposed in [6]).

In contemporary logic the appropriate setting for accommodating all such systems are the *multiple-conclusion logics* (also called *Scott consequence relations*, see [2,34]). Multiple-conclusion logics (or multiple-deductive systems) allow for consequence relations between sets of formulas, as $\Gamma \vdash \Delta$ with the intended meaning that under the assumptions Γ at least one element of Δ holds; in particular, this type of consequence

[5] This method of refutation is also imparted in Aristotle's treatise *De Sophisticis Elenchis*.

relations permit to express the exact duality between conjunction and disjunction;[6] indeed, while the pairs of rules:

$$\varphi \wedge \psi \vdash \varphi, \quad \varphi \wedge \psi \vdash \psi \quad \text{and} \quad \varphi \vdash \varphi \vee \psi, \quad \psi \vdash \varphi \vee \psi$$

express half of the intended $\wedge - \vee$ duality, the other half can only be expressed permitting sets (instead of singletons) at the right side of consequence relations:

$$\varphi, \psi \vdash \varphi \wedge \psi \quad \text{and} \quad \varphi \vee \psi \vdash \varphi, \psi$$

under the intended meaning prescribed above.

Before defining our general notion for dualizing logics we recall some concepts.

Definition 2.2. A (*multiple*) *deductive system* S consists of a pair $(L, \vdash_S)$ where $\vdash_S$ is a consequence relation defined on $\wp(\mathit{Sent}(L)) \times \wp(\mathit{Sent}(L))$ satisfying the following properties:

(1) *Reflexivity*: $\varphi \vdash_S \varphi$ for every formula φ;
(2) *Transitivity* or *Cut*: If $\Gamma_1 \vdash_S \Delta_1, \varphi$ and $\varphi, \Gamma_2 \vdash_S \Delta_2$ then $\Gamma_1, \Gamma_2 \vdash_S \Delta_1, \Delta_2$;
(3) *Monotonicity* or *Weakening*: If $\Gamma_1 \vdash_S \Delta_1$, $\Delta_1 \subseteqq \Delta_2$ and $\Gamma_1 \subseteqq \Gamma_2$ then $\Gamma_2 \vdash_S \Delta_2$;
(4) *Compactness* or *Finiteness*: If $\Gamma_1 \vdash_S \Delta_1$ then there exist finite Γ_0, Δ_0 such that $\Gamma_0 \vdash_S \Delta_0$;
(5) *Structurality*: If $\Gamma \vdash_S \Delta$ then $\sigma(\Gamma) \vdash_S \sigma(\Delta)$ for every (uniform) substitution σ, where

$$\sigma(\Theta) := \{\sigma(\theta): \theta \in \Theta\}.$$

Of course, a multiple deductive system reduces to single deductive system whenever the right side is composed of singletons, and the usual notion of derivability (for monotonic, compact and uniform logics) is just a particular case of this definition. Although single deductive systems are sufficient for most of our logics, it is convenient to explain all notions in terms of multiple-conclusions, as part of the objectives of this paper is to define a general setting for exploring logical duality.

Sometimes we do not have a deductive system for a logic, but just a semantical stipulation; we define now the concept of logical matrix, cf. [15], and valuation with respect to a logical matrix.

Definition 2.3. Let L be the language introduced before. A *logical matrix* $\mathfrak{M}$ is a pair $(\mathcal{T}, D)$ such that $\mathcal{T}$ is a non-empty algebra given by $(T, \wedge_M, \vee_M, \neg_M, \to_M, \bot_M)$ with the three binary operations $\wedge_M, \vee_M, \to_M$, the unary operation $\neg_M$ and one 0-ary operation $\bot_M$ and D is a non-empty proper subset of the carrier set T (of *truth-values*), called set of *designated truth-values*.

Definition 2.4. Given a logical matrix $\mathfrak{M} := (\mathcal{T}, D)$, we define a *valuation* v *with respect to the logical matrix* $\mathfrak{M}$ as a function $v : \mathcal{P} \to T$, where $\mathcal{P}$ is the set of the proposition

[6] We thank João Marcos for the definitions involving multiple-conclusions; our definition of duality is taken from [21].

letters, extended by induction on the complexity:

$$[\bot] \quad v(\bot) := \bot_M \in T,$$

$$[\neg] \quad v(\neg\varphi) := \neg_M v(\varphi), \qquad [\wedge] \quad v(\varphi \wedge \psi) := \wedge_M\big(v(\varphi), v(\psi)\big),$$

$$[\vee] \quad v(\varphi \vee \psi) := \vee_M\big(v(\varphi), v(\psi)\big), \qquad [\to] \quad v(\varphi \to \psi) := \to_M\big(v(\varphi), v(\psi)\big).$$

We denote by $\mathcal{V}_{\mathfrak{M}}$ the set of all such valuations v with respect to $\mathfrak{M}$.

We are now able to introduce the following notion:

Definition 2.5. Given a logical matrix $\mathfrak{M}$, we define a *semantical deductive system S with respect to* $\mathfrak{M}$ as a pair $S := (L, \vdash_S)$, such that $\vdash_S$ is a binary relation on $\wp(\mathit{Sent}(L)) \times \wp(\mathit{Sent}(L))$ satisfying for $\Gamma \cup \Delta \subseteq \mathit{Sent}(L)$:

$$\Gamma \vdash_S \Delta \quad \text{iff} \quad \text{for every valuation } v \in \mathcal{V}_{\mathfrak{M}} \text{ if } v(\gamma) \in D \text{ for all } \gamma \in \Gamma,$$
$$\text{then } v(\varphi) \in D \text{ for some } \varphi \in \Delta.$$

In the sequel we simply refer to a *logical system S* when it is irrelevant whether we are using a semantical deductive system or deductive systems, and use the notation $\vdash_S$ for both, when there is no danger of confusion. For example, we deal with a deductive system for Heyting's calculus, and with semantical deduction for Gödel's Calculi.

In both Definitions 2.2 and 2.5 the consequence relations satisfy the well-known Tarski's clauses for consequence relations in the case of single deductive systems (cf. [36]) and the Scott's axioms in the case of multiple deductive systems (cf. [32]).

We now introduce the notion of dual of a logical system, with the consequence that, when a deductive system is available, it is also possible to obtain a counter-deductive system with counter-axioms and counter-inference rules by means of the translation $*$ applied to the axioms and the inference rules of the given deductive system. We will see this in detail in next section. In the case of a semantical deductive system we introduce the following:

Definition 2.6. (a) Given a logical matrix $\mathfrak{M}$ as in Definition 2.3, we define the *dual logical matrix of* $\mathfrak{M}$, denoted by $\mathfrak{M}^*$, as a pair $(\mathcal{T}^*, D^*)$ such that

(i) $\mathcal{T}^* := (T, \wedge_{M^*}, \vee_{M^*}, \neg^*_{M^*}, -_{M^*}, \top_{M^*})$ is a non-empty algebra with the three binary operations $\wedge_{M^*}, \vee_{M^*}, -_{M^*}$, the unary operation $\neg^*_{M^*}$ and one 0-ary operation $\top_{M^*}$ defined as follows: Let $t, s \in T$

$$[\top] \quad \top_{M^*} := \bot_M,$$

$$[\neg^*] \quad \neg^*_{M^*}(t) := \neg_M(t), \qquad [\wedge^*] \quad \wedge^*_{M^*}(t, s) := \vee_M(t, s),$$

$$[\vee^*] \quad \vee^*_{M^*}(t, s) := \wedge_M(t, s), \qquad [-] \quad -_{M^*}(t, s) := \to_M(s, t), \quad \text{and}$$

(ii) $D^* = D \subset T$ is a non-empty proper subset of counter-designated values.

(b) Given a logical matrix $\mathfrak{M}$ and a valuation $v \in \mathcal{V}_{\mathfrak{M}}$, then we define the *dual valuation v^* with respect to the dual logical matrix* $\mathfrak{M}^*$ by the dual valuation function

$$v^* : \mathit{Sent}(L^*) \to T \quad \text{defined by} \quad v^*(\varphi^*) := v(\varphi).$$

Let us observe that if $v(\varphi) \in D$, then $v^*(\varphi^*) \in D^*$ is considered to be counter-designated and therefore must be rejected. It is worth noting that a minimal element of T is maximal in T^* and *vice versa*. Furthermore, for the purposes of v^*, we denote all values in $T - D$ as *designated* values. With these definitions we are now able to introduce the concept of a *dual logical system*.

Definition 2.7. Given a logical system $S := (L, \vdash_S)$ we define the *dual logical system* as $S^* := (L^*, \dashv_{S^*})$, where L^* is the previously introduced dual language, satisfying for $\Gamma \cup \Delta \subseteq Sent(L)$

$$\Gamma^* \dashv_{S^*} \Delta^* \quad \text{iff} \quad \Gamma \vdash_S \Delta.$$

Clearly, dual logical systems satisfy the Tarskian and Scottian conditions as in [32,36]. Furthermore, if a logical system is characterized syntactically and semantically by means of a completeness theorem, then completeness characterization also holds for the dual calculus.

In intuitive terms, by dualizing a logical system S we proceed in an elenctic manner, rejecting conjectures instead of proving new theorems. On the other hand, we are interested also in deictic dual logical systems, which allows us to positively produce theorems. In the following we are going to introduce such systems semantically and syntactically, by means of multiple deductive systems.

Definition 2.8. (a) Let $S := (L, \vdash_S)$ be a semantical deductive system with respect to a logical matrix and let its dual be $S^* := (L^*, \dashv_{S^*})$; we define the *deictic dual semantical deductive system* as the pair $S^* := (L^*, \vdash_{S^*})$ such that $\vdash_{S^*}$ is a binary relation on $\wp(Sent(L^*)) \times \wp(Sent(L^*))$ satisfying for $\Gamma^* \cup \Delta^* \subseteq Sent(L^*)$ (where Γ^*, Δ^* are as in Definition 2.7):

$\Gamma^* \vdash_{S^*} \Delta^*$ iff for every dual-valuation v^* in the valuation semantics which assigns a designated value to all sentences of Γ^* also attributes a designated value to some $\varphi^* \in \Delta^*$.

(b) Let $S := (L, \vdash_S)$ be a deductive system and let $S^* := (L^*, \dashv_{S^*})$ be its dual; we define the *deictic dual deductive system* as the pair $S^* := (L^*, \vdash_{S^*})$ such that $\vdash_{S^*}$ is a binary relation on $\wp(Sent(L^*)) \times \wp(Sent(L^*))$ satisfying for $\Gamma^* \cup \Delta^* \subseteq Sent(L^*)$ and nonempty Γ:

$$\Gamma^* \vdash_{S^*} \Delta^* \quad \text{iff} \quad \Delta^* \dashv_{S^*} \Gamma^*.$$

It is to be remarked that in both cases of Definition 2.8 we have:

$$\Gamma^* \vdash_{S^*} \Delta^* \quad \text{iff} \quad \Delta \vdash_S \Gamma$$

comprising, together with Definition 2.7, an abstract characterization of logical duality.

The notion of a deictic dual deductive system is well exemplified in the case of the hierarchy I^{n*}, $n \in \omega$, to be introduced in Section 5, where the dual deductive system coincides with $\vdash_{P^n}$.

We are able now to introduce what we mean by a *self-dual* logical system.

Definition 2.9. With the above notations, let $S := (L, \vdash_S)$ be a semantical deductive system. We say that S is *self-dual* iff

$$\vdash_S \varphi \quad \text{iff} \quad \vdash_{S^*} \neg^* \varphi^*. \tag{$*$}$$

A simple argument shows that classical logic is in this sense self-dual, a very much natural result. Fragments of classical logic (for example, the conjunctive fragment, the disjunctive fragment and the conjunctive–disjunctive fragment) will also be self-dual. We do not know how to characterize self-dual logics in general: as it will be seen in the following, the intuitionistic and paraconsistent logics $I^n P^n$, $n \in \omega$ mentioned in [16] are not self-dual in our sense, although $I^n P^n$ coincide with their duals in the sense that their deictic valuation semantics coincide, after introducing an internal implication symbol. We also see in the next section that Johansson's calculus is not self-dual, although being intuitionistic and paraconsistent.

In the following we will be interested in dualizing intuitionistic logics in a broad sense; it will be shown that the dual of intuitionistic logics are always paraconsistent logics.

3. Heyting's and Johansson's dual calculi

3.1. The dual of Heyting's calculus

We begin by investigating Heyting's dual calculus. In [19], Heyting's dual calculus was treated considering a dual sequent system in the sense of Gentzen. The semantics is given as algebraic models using methods of Rasiowa and Sikorski [29]. In [37], Heyting's dual calculus is treated syntactically, formulating Gentzen's sequent system in various manners.

Our approach here, as in [30], will be to give a elenctic (i.e., refutative) Hilbert-type axiomatization of Heyting's dual calculus H^*. We provide a list of counter-axioms plus a counter-inference rule, which permit to assort the sentences which are everywhere invalid. The logic H^* is a liberal one, in the sense that we accept almost anything in the beginning and subsequently advance by rejecting sentences which are patently false. Instead of proving new theorems, in our anti-intuitionistic logics we are going to reject new conjectures in a typical elenctic style. Using the dualizing translation $*$ introduced before, we obtain the following system of counter-axioms:

Let φ, ψ and χ be formulas, then

I. $(\varphi - \psi) - \varphi$;
II. $[(\chi - \varphi) - ((\chi - \psi) - \varphi)] - (\psi - \varphi)$;
III. $[(\varphi \vee \psi) - \varphi] - \psi$;
IV. $\varphi - (\psi \vee \varphi)$; $\psi - (\psi \vee \varphi)$;
V. $(\varphi \wedge \psi) - \varphi$; $(\varphi \wedge \psi) - \psi$;
VI. $[(\chi - (\varphi \wedge \psi)) - (\chi - \varphi)] - (\chi - \psi)$;
VII. $\varphi - \top$.

Counter-Inference Rule:

$$[\text{Dual Modus Ponens}] \quad \frac{\varphi, \psi - \varphi}{\psi}.$$

We use $\Gamma \dashv_{H^*} \varphi$ to denote that if all formulas of Γ are rejected, then φ is also rejected. By writing $\dashv_{H^*} \psi$ we mean that ψ is a counter-theorem or equivalently, that ψ is rejected from the counter-axioms H^*.

In the following we will obtain (as in [5,30,31]) a Kripke semantics for this system of counter-axioms which we prove to be sound and complete.

Definition 3.1. The *anti-intuitionistic Kripke model* is a triple $\mathcal{K} := (K, \leqslant, \Vdash_{H^*})$ where $(K, \leqslant)$ is a partially ordered set (poset) and $\Vdash_{H^*}$ is a binary relation on $K \times \mathcal{P}$ (where $\mathcal{P}$ is the set of proposition letters) such that, for every $k \in K$ and $p \in \mathcal{P}$:

$$\exists k' \geqslant k \quad k' \Vdash_{H^*} p \quad \Longrightarrow \quad k \Vdash_{H^*} p.$$

We say, if $k \Vdash_{H^*} p$, that k *forces anti-intuitionistically* p.

The notion of semantical forcing is extended to logically compound formulas through the following clauses:

$\underline{[\wedge]}$ $\quad k \Vdash_{H^*} \varphi \wedge \psi \quad$ iff $\quad k \Vdash_{H^*} \varphi$ and $k \Vdash_{H^*} \psi$.

$\underline{[\vee]}$ $\quad k \Vdash_{H^*} \varphi \vee \psi \quad$ iff $\quad k \Vdash_{H^*} \varphi$ or $k \Vdash_{H^*} \psi$.

$\underline{[-]}$ $\quad k \Vdash_{H^*} \varphi - \psi \quad$ iff $\quad \exists k' \geqslant k$ ($k' \Vdash_{H^*} \varphi$ and $k' \nVdash_{H^*} \psi$).

$\underline{[\top]}$ $\quad k \Vdash_{H^*} \top \quad \forall k \in K$.

One can define an anti-intuitionistic negation as $\neg^* \varphi := \top - \varphi$, and it can be easily shown that:

$$k \Vdash_{H^*} \neg^* \varphi \quad \text{iff} \quad \exists k' \geqslant k \; k' \nVdash_{H^*} \varphi.$$

It is worth remarking that the operation of rejecting sentences is monotonic, as can be proven by induction on complexity:

$$k \nVdash_{H^*} \varphi \quad \Longrightarrow \quad \forall k' \geqslant k \; k' \nVdash_{H^*} \varphi.$$

This property lays the groundwork for our considerations that the anti-intuitionistic logic is endowed with a elenctic character in the sense that it rejects new conjectures instead of demonstrating theorems.

Finally, it is also important to point out that our approach permits to give an algebraic valuation semantics at once, as done in [24], just by dualizing the Heyting algebra (obtaining thus a Brouwerian algebra). Also, it is clear that our Kripke model is obtained by a simple dualization of a usual Kripke model. Observe that our consequence relation introduced in Definition 3.1 is in the sense of Definition 2.8 a deictic dual semantical consequence relation.

In what follows we show, illustrating by an example, that the anti-intuitionistic logic H^* is in effect paraconsistent.

Definition 3.2. A formula φ is *valid at the world* k of an anti-intuitionistic Kripke model $\mathcal{K}$ iff $k \Vdash_{H^*} \varphi$. A formula φ is *valid in* $\mathcal{K}$, denoted by $\mathcal{K} \Vdash_{H^*} \varphi$, iff for all $k \in K$, $k \Vdash_{H^*} \varphi$. For a set Γ of sentences, we say that $\Gamma \Vdash_{H^*} \varphi$ (i.e., φ is a Kripke consequence of Γ) iff for each model $\mathcal{K}$ and each $k \in K$ we have that if $k \Vdash_{H^*} \gamma$ for all $\gamma \in \Gamma$, then also $k \Vdash_{H^*} \varphi$. A sentence φ is said to be *Kripke-valid* iff $\emptyset \Vdash_{H^*} \varphi$.

Example 3.3. Consider now the following anti-intuitionistic Kripke model:

$$\begin{tikzcd} 1;\, p & & 2;\, \neg^* p \\ & 0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

where p and q are propositional letters. It is easy to see that $0 \Vdash_{H^*} p \wedge \neg^* p$, because $1 \Vdash_{H^*} p$ and $2 \Vdash_{H^*} \neg^* p$. Furthermore, $0 \nVdash_{H^*} q$. Consequently, the Principle of Explosion or of Pseudo-Scotus (also known as *ex contradictione sequitur quodlibet*, see [10] for a discussion) does not hold, and this Kripke semantics is indeed paraconsistent (i.e., $\Vdash_{H^*}$ is not explosive in the sense of Definition 3.13).

On the other side, it is easily proven that the law of *tertium non datur* is valid in our Kripke semantics, that is, $\Vdash_{H^*} \varphi \vee \neg^* \varphi$.

The next theorems are an easy consequence of the dual character of our logics and of soundness and completeness for the original Heyting calculus H.

Theorem 3.4 (Soundness). *There is no Kripke model $\mathcal{K}$ and no $k \in K$ such that $k \Vdash_{H^*} \varphi$ for any counter-theorem φ of H^*.*

Equivalently, if $\dashv_{H^*} \varphi$ then for all $\mathcal{K}$ and all $k \in K$ we have $k \nVdash_{H^*} \varphi$.

Theorem 3.5 (Completeness). *If for all $\mathcal{K}$ and all $k \in K$, $k \nVdash_{H^*} \psi$, then $\dashv_{H^*} \psi$, or equivalently, if $\not\dashv_{H^*} \psi$, then there is a Kripke model $\mathcal{K}$ and $k \in K$ such that $k \Vdash_{H^*} \psi$.*

Remark 3.6. The result of Theorem 3.5 also applies to strong completeness, formulated in the following form:

$$\Gamma \dashv_{H^*} \varphi \quad \text{iff} \quad \forall \mathcal{K}(\mathcal{K} \Vdash_{H^*} \gamma \text{ for some } \gamma \in \Gamma \text{ or } \mathcal{K} \nVdash_{H^*} \varphi).$$

In [19] a soundness and completeness result for H^* was established considering algebraic models in the sense of Rasiowa and Sikorski. It is to remark here (and this is not difficult to prove, just dualize the proof given in [17]) that our Kripke models are equivalent to such algebraic models with values in Brouwerian algebras, and vice versa. Also, applying Theorem 15 in [19] we can immediately infer the completeness for H^* with respect to the dualized semantics. From this it follows that Goodman's sequent calculus is equivalent to our elenctic Hilbert style calculus. Furthermore, in [19,37], it is proven that there does not exist an internal implication symbol in the sense of [1]:

Definition 3.7. Let $S := (L, \vdash_S)$ be a logical system in the sense of Section 2. We say that a binary connective $\rightarrow^*$ is an *internal implication symbol* iff for $\Gamma \cup \{\varphi, \psi\} \subseteq Sent(L)$ we have the following property

$$\Gamma, \varphi \vdash_S \psi \quad \text{iff} \quad \Gamma \vdash_S \varphi \to^* \psi.$$

Using our system of counter-axioms H^*, it can be easily proven that

$$\varphi \vee \neg^* \varphi \dashv_{H^*} \psi \quad \forall \psi \in Sent(L^*).$$

That is, a tautology "counter-trivializes" our system H^* in the sense that, in the presence of a tautology, one must reject every sentence.

On the other side, it is easily seen (e.g., by applying the Completeness Theorem) that

$$\varphi \wedge \neg^* \varphi \not\dashv_{H^*} \psi \quad \forall \psi \in Sent(L^*).$$

3.2. The dual of Johansson's calculus

The same dualization procedure is applicable to Johansson's calculus J, which is convenient to consider here as intuitionistic and paraconsistent, due to the fact that Johansson (cf. [20]) did not accept the principle of $\bot \to \varphi$ as constructive. Because Johansson's calculus is a very basic one from the point of view of intuitionism, we are interested in its dual J^* and in the questions related to its self-duality, if it is to be regarded as paraconsistent, we will see anyhow that it is not self-dual.

Firstly, it is clear that we can immediately obtain, by the same maneuver as above, a Hilbert type system of counter-axioms for J^* considering the counter-axioms (I) to (VI) of H^* and the counter-inference rule of Dual Modus Ponens. Observe that the absence of counter-axiom (VII) has as a consequence that a tautology does not counter-trivialize J^* in any way. Furthermore, we are able to define a Kripke semantics for J^* using the same postulates as for H^*, except the $[\top]$-postulate. We will denote this forcing relation by $\Vdash_{J^*}$, defining it as follows:

Definition 3.8. The *anti-intuitionistic Kripke model for Johansson's dual calculus* is a triple $\mathcal{K} := (K, \leqslant, \Vdash_{J^*})$ where $(K, \leqslant)$ is a partially ordered set (poset), and $\Vdash_{J^*}$ is a binary relation on $K \times \mathcal{P}$ ($\mathcal{P}$ is the set of proposition letters) such that, for every $k \in K$ and $p \in \mathcal{P}$:

$$\exists k' \geqslant k \quad k' \Vdash_{J^*} p \quad \Longrightarrow \quad k \Vdash_{J^*} p.$$

We say if $k \Vdash_{J^*} p$ that k *forces anti-intuitionistically* p.

The notion of semantical forcing is extended to logically compound formulas through the following clauses:

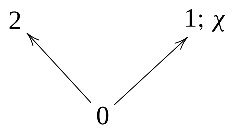
$\underline{[\wedge]}$ $\quad k \Vdash_{J^*} \varphi \wedge \psi \quad$ iff $\quad k \Vdash_{J^*} \varphi$ and $k \Vdash_{J^*} \psi$.

$\underline{[\vee]}$ $\quad k \Vdash_{J^*} \varphi \vee \psi \quad$ iff $\quad k \Vdash_{J^*} \varphi$ or $k \Vdash_{J^*} \psi$.

$\underline{[-]}$ $\quad k \Vdash_{J^*} \varphi - \psi \quad$ iff $\quad \exists k' \geqslant k \ (k' \Vdash_{J^*} \varphi$ and $k' \nVdash_{J^*} \psi)$.

Soundness and completeness for this calculus follow from the same dual considerations as for H^*. Thus, derivability in J^* can in principle be treated by syntactical or by semantical means. To what concerns the status of J^*, we argue below that J^* is in effect paraconsistent and intuitionistic, besides being non-self-dual.

Example 3.9. To show that J^* is paraconsistent, it suffices to recall Example 3.3. For showing that J^* has an intuitionistic complexion consider the following situation:



In the above model none of the node forces ψ and also none of the node forces $\top$. Consequently, $0 \nVdash_{J^*} \psi \vee \neg^*\psi$ and J^* refutes the law of excluded middle.

Recalling that in Johansson's calculus J the following is valid:

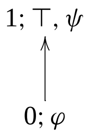
$$\vdash_J (\varphi \wedge \neg\varphi) \to \neg\psi \quad \text{and} \quad \nvdash_J (\varphi \wedge \neg\varphi) \to \psi$$

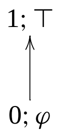
for the dual calculus J^* we attain the following:

Lemma 3.10. *Let φ, ψ be formulas in L^*. Then*:

(a) $\varphi \wedge \neg^*\varphi \nVdash_{J^*} \neg^*\psi$ *and* $\varphi \wedge \neg^*\varphi \nVdash_{J^*} \psi$.
(b) $\varphi \wedge \neg^*\varphi \Vdash_{J^*} \psi \vee \neg^*\psi$.
(c) $\vdash_J \varphi \to (\psi \to \varphi)$, *but not* $\Vdash_{J^*} \neg^*((\varphi^* - \psi^*) - \varphi^*)$.

Proof. (a) Consider the following models:





Then the first model gives an example for the first part of the claim. It is clear that $0 \Vdash \varphi \wedge \neg^*\varphi$ and $0 \nVdash \neg^*\psi$. The second example finishes the proof of item (a).

(b) Let $\mathcal{K}$ be an arbitrary anti-intuitionistic Kripke model and suppose that $k \Vdash_{J^*} \varphi \wedge \neg^*\varphi$. By definition, this means that:

$$k \Vdash_{J^*} \varphi \quad \text{and} \quad \exists k' \geqslant k\ (k' \Vdash_{J^*} \top \text{ and } k' \nVdash_{J^*} \varphi). \tag{$*$}$$

Remember that $k \Vdash_{J^*} \psi \vee \neg^*\psi$ means that

$$k \Vdash_{J^*} \psi \quad \text{or} \quad \exists k' \geqslant k\ (k' \Vdash_{J^*} \top \text{ and } k' \nVdash_{J^*} \psi).$$

To finish the proof, we only have to examine two cases. If $k \Vdash_{J^*} \psi$, then we are done. If not, then by $(*)$ there is a $k' \geqslant k$ such that $k' \Vdash_{J^*} \top$ and $k' \nVdash_{J^*} \psi$, by the monotonicity of rejection.

(c) A simple exercise of applying Definition 3.8 and considering that not necessarily every node forces $\top$. □

One obtains therefore, as a direct consequence:

Corollary 3.11. *Johansson's calculus J is not self-dual.*

Due to the fact that there are models for Johansson's dual calculus J^* which do not validate $\top$, we have as an immediate consequence that J^* *cannot* validate negated tautologies. Therefore, J^* looks, in a certain sense, similar to the positive fragment of classical logic; this property of J^* is shared with the paraconsistent logics $C_{\min}$ and C_ω (cf. [10], Theorem 3.5). In accordance with this, the law of non contradiction cannot be valid in J^*. A counter-model is the following:

Example 3.12. We claim that $\not\Vdash_{J^*} \neg^*(\varphi \wedge \neg^*\varphi)$. Let us consider the following counterexample:

$$\begin{tikzcd} 1; \varphi \\ 0; \varphi \arrow[u] \end{tikzcd}$$

If $0 \Vdash_{J^*} \neg^*(\varphi \wedge \neg^*\varphi)$, then by definition there must be $k \geqslant 0$ such that $k \Vdash_{J^*} \top$ and $k \not\Vdash_{J^*} \varphi \wedge \neg^*\varphi$, which is not the case in the above model.

There are several questions about dual intuitionistic logics: is there a *strong* negation symbol in H^* and J^*, in the sense of [10]? And if there is, is that negation a *classical* negation? Furthermore, is it possible to rewrite these two logics as *logics of formal inconsistency* in the sense of [10]? In special, is it possible to devise semantics of possible-translations for them (in the sense of [7], see also [11])? Is there an internal implication symbol to be definable in J^*? We will see that the first two questions have an affirmative answer, while all others remain open.

Let us begin with a few definitions, repeated from [10].

Definition 3.13. Let $S := (L, \vdash_S)$ be a deductive system. Then

(a) S is called *explosive* if $\neg\varphi, \varphi \vdash_S \psi\ \forall\psi \in \mathit{Sent}(L)$.
(b) S has a *bottom particle* if there is a formula $\perp$ in L such that $\perp \vdash_S \psi\ \forall\psi \in \mathit{Sent}(L)$.
(c) S has a *strong negation* if for every formula φ there is a formula $\sigma(\varphi)$ such that $\sigma(\varphi)$ is not a bottom particle and

$$\sigma(\varphi), \varphi \vdash_S \psi \quad \forall\psi \in \mathit{Sent}(L).$$

(d) Let $\sim$ denote a negation symbol; we say that $\sim$ is a *classical negation* in S if it satisfies for $\varphi \in \mathit{Sent}(L)$
 – $\vdash_S \varphi \vee \sim\varphi$,
 – $\sim\sim\varphi \vdash_S \varphi$, and
 – $\sim\varphi, \varphi \vdash_S \psi\ \forall\psi \in \mathit{Sent}(L)$.
(e) A theory Γ is *gently explosive* if there is an L-formula φ and a schema $\Delta(\varphi)$ such that $\Delta(\varphi) \cup \{\varphi\}$ and $\Delta(\varphi) \cup \{\neg\varphi\}$ are not trivial and $\Gamma, \Delta(\varphi), \varphi, \neg\varphi \vdash_S \psi\ \forall\psi \in \mathit{Sent}(L)$.
(f) S is called *gently explosive* if every theory is gently explosive.
(g) S is called a *logic of formal inconsistency* if it is not explosive but gently explosive.

With these definitions we observe firstly that every counter-axiom of our anti-intuitionistic logics J^* and H^* defines a bottom particle $\perp$ in the above sense. Let us now define—in the same manner as in [10] in Fact 2.10(ii)—a strong negation symbol $\sim$:

(i) $\Vdash \sim\varphi$ iff $\varphi \Vdash \perp$, and
(ii) $\sim\varphi \Vdash \perp$ iff $\Vdash \varphi$,

where $\Vdash$ stands for the semantical deduction in H^* and J^*, i.e., $\Vdash_{H^*}$ and $\Vdash_{J^*}$, respectively. It is an easy exercise to show that $\sim$ is really a strong negation in J^* and H^*. Furthermore, this strong negation $\sim$ is also a classical negation in the above sense. Nevertheless, it remains still open whether or not J^* and H^* can be considered as logics of formal inconsistency, and whether an internal implication can be defined in J^*. We conjecture that the answer is negative in both cases, but cannot advance any argument.

It is plainly possible to extend such anti-intuitionistic logics H^* and J^* for the predicate level, as done in [37]. It is also possible to obtain for them a sound and complete Kripke semantics.

4. The hierarchy of Gödel's dual calculi

In [18], Gödel introduced the hierarchy $(G_n)_{n\geqslant 2}$ of many-valued intuitionistic logics by means of appropriate valuation semantics. We study in the sequel the duals of this hierarchy applying the dualization method developed in Section 2, obtaining a hierarchy of anti-intuitionistic logics $(G_n^*)_{n\geqslant 2}$. We thus show that it is not possible to introduce any internal implication symbol for any calculus of this hierarchy.

Let $\mathcal{P}$ denote the proposition letters; then define a G^**-valuation* as an application $e:\mathcal{P}\to[0;1]$, which is extendable by induction on the complexity in the following way:

$$e(\top) := 1, \qquad e(\varphi\wedge\psi) := \min\bigl\{e(\varphi); e(\psi)\bigr\}, \qquad e(\varphi\vee\psi) := \max\bigl\{e(\varphi); e(\psi)\bigr\}$$

and

$$e(\varphi - \psi) := \begin{cases} 0 & \text{if } e(\varphi)\leqslant e(\psi);\\ e(\varphi) & \text{otherwise.}\end{cases}$$

For $n\geqslant 2$, we define:

$$G_n^* := \left\{\psi \mid e(\psi)\neq 0 \text{ for every } G^*\text{-valuation } e:\mathcal{P}\to\left\{\frac{m}{n-1} \mid 0\leqslant m\leqslant n-1\right\}\right\}.$$

$$G_{\aleph_0}^* := \bigl\{\psi \mid e(\psi)\neq 0 \text{ for every } G^*\text{-valuation } e \text{ which takes only rational values in } [0;1]\bigr\}.$$

Notice that for the anti-intuitionistic negation we have then:

$$e(\neg^*\psi) = \begin{cases} 0 & \text{if } e(\psi)=1;\\ 1 & \text{if } e(\psi)\neq 1.\end{cases}$$

The above G^*-valuations arise by dualization, in the sense of Section 2, of the valuation semantics for the Gödel Calculi G_n, given in [15]. Therefore, the undesignated value is the value 0, while the designated values are all non-zero values of the above interval.

Definition 4.1. Let φ be a formula in L^*. Then, we define the following

(a) φ is a *counter-tautology in* G_n^* if $e(\varphi) = 0$ for every valuation e, i.e., $\dashv_{G_n^*} \varphi$.
(b) φ is a *tautology* G_n^*, if $e(\varphi) \neq 0$ for every valuation e, i.e., $\vdash_{G_n^*} \varphi$.
(c) A sentence ψ is an *anti-intuitionistic consequence in* G_n^* of a set of formulas Γ, $\Gamma \vdash_{G_n^*} \psi$, iff for every valuation e, if $e(\gamma) \neq 0$ for all $\gamma \in \Gamma$ then we have $e(\psi) \neq 0$.

Remark 4.2. Recall that $\vdash_{G_n^*}$ is exactly the binary relation defined in Definition 2.8 item (a), and $\dashv_{G_n^*}$ is the binary relation defined in Definition 2.7.

For reasons of simplicity we treat here only the calculus G_3^*; a simple induction will produce similar results for the other Gödel Calculi. A straightforward calculation shows that the formulas in $G^* := H^* \cup \{(\varphi - \psi) \wedge (\psi - \varphi)\colon \varphi, \psi \in L^*\}$, where H^* denotes the set of the counteraxioms I–VII in Section 3, are all counter-tautologies in G_3^*. More than this, by dualizing a result of Dummett (cf. [14]) we can easily show that $G_{\aleph_0}^*$ is sound and complete with respect to the counter-axioms of G^*. The next example establishes that our newly obtained hierarchy is paraconsistent and satisfies the law of excluded middle.

Example 4.3. It is easy to see that $e(\varphi \vee \neg^*\varphi) \neq 0$ for every valuation e. If $e(\varphi) = 1$, then $e(\varphi \wedge \neg^*\varphi) = 0$ and this shows that $\varphi \wedge \neg^*\varphi$ is not a tautology. But there are valuations e such that $e(\varphi \wedge \neg^*\varphi) \neq 0$, which shows that in G_3^* we can have contradictory situations. The next result shows that this calculus is not trivial (albeit contradictory): Let φ and ψ be formulas such that $e(\varphi) \neq 1$ and $e(\psi) = 0$, then $e(\varphi \wedge \neg^*\varphi) = e(\varphi)$ is distinguished but ψ is not deducible from this contradiction. Therefore, $\varphi \wedge \neg^*\varphi \nvdash_{G_3^*} \psi$ and G_3^* is paraconsistent. On the other side, $\neg^*(\varphi \wedge \neg^*\varphi)$ is always valid. Also, it can be easily seen that $\neg^*\neg^*\varphi \vdash_{G_3^*} \varphi$. For the converse, it is clear that for $e(\varphi) = \frac{1}{2}$, we have $e(\neg^*\neg^*\varphi) = 0$, which shows that $\varphi \nvdash_{G_3^*} \neg^*\neg^*\varphi$.

Remark 4.4. From the fact that $\varphi \vee \neg\varphi$ is not a tautology in the Gödel Calculi G_n, and the fact that $\neg^*(\varphi \wedge \neg^*\varphi)$ is a tautology in G_n^* it immediately follows that G_n are not self-dual.

The next proposition shows that the tautologies in G_3^* coincide with the classical tautologies.

Proposition 4.5. *Interpreting in* G_3^* *a formula* $\varphi - \psi$ *by* $\varphi \wedge \neg^*\psi$ *we have the following equivalence*:

$$\varphi \textit{ is a tautology in } G_3^* \quad \textit{iff} \quad \varphi \textit{ is a classical tautology}.$$

Proof. One direction is clear. For the other, define for a propositional letter p, where v is a classical valuation:

$$v(p) = \top \quad \text{iff} \quad e(p) = 1.$$

By induction on the complexity, we can show the following:

(i) $v(\varphi) = \top \Rightarrow e(\varphi) = 1$.
(ii) $v(\varphi) = \bot \Rightarrow e(\varphi) \in \{0, \frac{1}{2}\}$.

The only non-trivial step is the negation case. Observe that:

$$v(\neg^* \varphi) = \top \quad \Leftrightarrow \quad v(\varphi) = \bot \quad \Rightarrow \quad \text{(ii) } e(\varphi) \in \left\{0, \tfrac{1}{2}\right\}$$

and therefore, $e(\neg^* \varphi) = 1$. For the case $v(\neg^* \varphi) = \bot$ just use induction hypothesis (i).

Let φ be a classical tautology, then we have, in fact, $e(\varphi) = 1$ for every valuation in G_3^*. Therefore, φ is a tautology in G_3^*. □

Corollary 4.6. *There is no internal implication in G_3^*.*

Proof. Suppose that an internal implication $\rightarrow^*$ were definable in G_3^*. In this case, we could define this symbol in classical logic. From $\varphi \vdash_{G_3^*} \varphi$ we obtain the validity of $\varphi \rightarrow^* \varphi$ in G_3^* and in classical logic. But $\varphi \rightarrow^* \neg^*\neg^*\varphi$ is a tautology in classical logic, and by the last proposition also in G_3^*. But this leads to $\varphi \vdash_{G_3^*} \neg^*\neg^*\varphi$, contradicting Example 4.3. □

In the same way as in Section 3 it is possible to introduce a strong negation symbol in Gödel's anti-intuitionistic hierarchy, which works also as a classical negation. Again, what remains open is whether or not the logics in this hierarchy are logics of formal inconsistency; again, we conjecture that they are not.

A simple generalization by induction establishes the following hierarchy of Gödel anti-intuitionistic logics, where G_2^* coincides with the classical logic:

$$G_{\aleph_0}^* \subsetneq \cdots \subsetneq G_n^* \subsetneq \cdots \subsetneq G_3^* \subsetneq G_2^*.$$

The hierarchy is proper (i.e., inclusions are not equalities). To see this, consider a language with an infinite number of propositional letters p_i, $i \in \omega$; thus each sentence:

$$\psi_n := \bigwedge_{1 \leqslant i < k \leqslant n+1} \left[(p_i - p_k) \vee (p_k - p_i)\right]$$

is a counter-theorem of G_n^*, but not of G_{n+1}^*.

5. The duals of the paracomplete logics $(I^n)_{n\in\omega}$

In this section we investigate the dualization of the hierarchy of the maximal weakly-intuitionistic logics (or paracomplete) $(I^n)_{n\in\omega}$, introduced for the three-valued case in [33], defined for the general case in [8] and further developed in [16].

In [33] is claimed, arguing intuitively, that I^1 is the first step of a hierarchy of duals of the paraconsistent calculus P^1; we will see in the following that this is indeed the case: I^{n*} is identical to P^n (see Corollary 5.5), if we introduce an internal implication symbol in the sense of Definition 3.7.

We start by dualizing the hierarchy $(I^n)_{n\in\omega}$, remembering that these n-valued logics are given by a valuation semantics. But there also exist Hilbert type axiomatizations for such logics, as obtained in [16]. To obtain the dualization we will, for simplicity, make use of the valuation semantics.

It is known (see [16]) that the connectives $\wedge$ and $\vee$ are definable from the negation and implication symbols. Therefore, by dualizing, the same will hold true in the dual calculi I^{n*}. Let $\mathcal{P}$ denote the proposition letters; then define an I^{n*}-valuation as an application

$$e:\mathcal{P}\to\left\{\tfrac{m}{n+1}\colon 0\leqslant m\leqslant n+1\right\},$$

which can be extended to all formulas by induction on the complexity, as follows:

$$e(\neg^*\psi)=\begin{cases}1 & \text{if } e(\psi)=0;\\ 0 & \text{if } e(\psi)=1;\\ \frac{m+1}{n+1} & \text{if } e(\varphi)\in\left\{\frac{m}{n+1}\colon 0<m<n+1\right\}.\end{cases}$$

$$e(\varphi-\psi):=\begin{cases}1 & \text{if } e(\psi)=0 \text{ and } e(\varphi)\in\left\{\frac{1}{n+1},\dots,1\right\};\\ 0 & \text{otherwise.}\end{cases}$$

By dualization, we can define conjunction and disjunction as:

$$\varphi\wedge\psi:=\psi-(\neg^*\varphi-\varphi)\quad\text{and}\quad\varphi\vee\psi:=\neg^*\bigl((\neg^*\psi-\psi)-\varphi\bigr).$$

Similar to the duals of the Gödel logics, and defining the designated values in I^{n*} as all values in $\{\frac{1}{n+1},\dots,1\}$, we have:

Definition 5.1. In the anti-intuitionistic calculi I^{n*}, $n\in\omega$, we define,

(a) A sentence φ is a *counter-tautology* iff for every valuation e, $e(\varphi)=0$, i.e., $\dashv_{I_n^*}\varphi$.
(b) A sentence φ is a *tautology* iff for every valuation e, $e(\varphi)\neq 0$, i.e., $\vdash_{I_n^*}\varphi$.
(c) A sentence φ is an *anti-intuitionistic consequence* of a set of sentences Γ, $\Gamma\vdash_{I^{n*}}\varphi$ iff for every valuation e, if $e(\gamma)\neq 0$ for all $\gamma\in\Gamma$, then $e(\varphi)\neq 0$.

With the above notation, the next proposition is easily provable and we omit the proof.

Proposition 5.2. *For $\Gamma\cup\{\varphi\}\subseteq Sent(L)$, we have that*

(a) *In the case, φ is not atomic, then $\vdash_{I^n}\neg\varphi$ iff $\vdash_{I^{n*}}\varphi^*$.*
For an atomic formula we do not have the direction $\Leftarrow$.
(b) *In case φ is not atomic, then $\vdash_{I^n}\varphi$ iff $\vdash_{I^{n*}}\neg^*\varphi^*$.*
For atomic formulas the direction $\Leftarrow$ does not hold.

This permits to obtain directly the following:

Corollary 5.3. *The logics of the hierarchy* $(I^n)_{n\in\omega}$ *are not self-dual.*

We observe that in the anti-intuitionistic calculi I^{n*} not all classical tautologies hold. For example, the classical law of the non contradiction $\neg^*(\varphi \wedge \neg^*\varphi)$ is not valid in I^{n*}. This shows also that these logics do not coincide with any logic of the hierarchy introduced in Section 4.

Our next problem regards the question about the existence of an internal implication symbol in I^{n*}. This question can be responded positively, as shown below.

Lemma 5.4. *In* I^{n*} *there is an internal implication symbol* $\to^*$ *in the sense of Definition 3.7 defined in the following way*: $\varphi \to^* \psi := \neg^*(\varphi - \psi)$.

Proof. By Definition 3.7 we have to show for $\Gamma \cup \{\varphi, \psi\} \subseteq Sent(L^*)$:

$$\Gamma, \varphi \vdash_{I^{n*}} \psi \quad \text{iff} \quad \Gamma \vdash_{I^{n*}} \varphi \to^* \psi.$$

This follows easily by calculating the valuation semantics for the defined internal implication symbol $\to^*$, which gives:

$$e(\varphi \to^* \psi) := \begin{cases} 0 & \text{if } e(\psi) = 0 \text{ and } e(\varphi) \in \left\{\frac{1}{n+1}, \dots, 1\right\}; \\ 1 & \text{otherwise.} \end{cases} \qquad \square$$

Corollary 5.5. *The anti-intuitionistic logics* I^{n*}, $n \in \omega$, *with the internal implication symbol* $\to^*$ *introduced in Lemma 5.4, coincide with the paraconsistent logics* P^n, $n \in \omega$, *in the common language, i.e., in the common language we have* $\vdash_{I^{n*}} = \vdash_{P^n}$.

Therefore, our method of dualization grants the definition, departing from $(I^n)_{n\in\omega}$, of a whole hierarchy of anti-weakly-intuitionistic (or anti-paracomplete) logics $(I^{n*})_{n\in\omega}$:

$$I^{1*} \supsetneq I^{2*} \supsetneq \cdots \supsetneq I^{n*} \supsetneq \cdots \tag{AP}$$

which we have shown to coincide with the paraconsistent hierarchy:

$$P^1 \supsetneq P^2 \supsetneq \cdots \supsetneq P^n \supsetneq \cdots \tag{P}$$

introduced in [33] and further developed in [16].

Applying this dualization method to the intuitionistic and paraconsistent calculi $(I^n P^m)_{n,m\in\omega}$, cf. [16], we obtain—omitting the details—the following results:

(i) $I^n P^m$, $n, m \in \omega$, are not self-dual in our sense.
(ii) Introducing in $I^n P^m$ internal implication symbols, we easily see that the dual of $I^n P^m$ coincide with $I^m P^n$ in the common language, i.e., we have in the common language $\vdash_{(I^n P^m)^*} = \vdash_{I^m P^n}$. In particular, we have that $\vdash_{(I^n P^n)^*} = \vdash_{I^n P^n}$ in the common language.

6. Additional properties and comparisons

6.1. The anti-constructive logics AC

In Sections 3 and 4 we have dualized the most known intuitionistic logics, which includes Heyting's calculus. These dualizations have led to the following hierarchy of anti-intuitionistic (or, anti-constructive) logics:

$$J^* \subsetneq H^* \subsetneq G^*_{\aleph_0} \subsetneq \cdots \subsetneq G^*_n \subsetneq \cdots \subsetneq G^*_3 \subsetneq G^*_2. \tag{AC}$$

This hierarchy *AC* of anti-intuitionistic logics has basically two main characteristics. The first characteristic is that, from a formula φ, we cannot deduce its double-negated formula. The second property consists of the *intersubstitutivity of provable equivalents*, that is, if we have any schema $\sigma(\varphi_1, \ldots, \varphi_n)$, and provable equivalents $\psi_1, \ldots, \psi_n$ such that φ_i is logically equivalent with ψ_i for $i = 1, \ldots, n$, then $\sigma(\varphi_1, \ldots, \varphi_n)$ is logically equivalent with $\sigma(\psi_1, \ldots, \psi_n)$. This second characteristic is very interesting, because several known paraconsistent logics fail to satisfy this intersubstitutivity property. There are other two characteristics of our hierarchy (excluding the dual calculus of Johansson J^*). The first is that the law of no contradiction $\neg^*(\varphi \wedge \neg^*\varphi)$ is valid in every dual calculus. The second is that all classical tautologies are also tautologies in the anti-intuitionistic logics, and vice versa. For that reason we have seen that it is impossible to introduce, in such cases, an internal implication symbol. As a consequence we must use semantics when we want to express implication. However, we do not agree with the observation in [19] that these anti-intuitionistic calculi are too weak to contain much mathematics: just recall, once again, that all classical tautologies are also tautologies in our hierarchies (with the exception of J^*). Also, thanks to the duality, we do not need an implication symbol: for example, if we prove $\varphi \to \psi$, then this is equivalent to rejecting $\psi^* - \varphi^*$.

Other interesting characteristics of the dualization method can be obtained by comparing original calculi with their duals. Some representative of this kind of properties are listed in the following remark.

Remark 6.1. The process of dualization from intuitionistic to anti-intuitionistic logic has the following properties. Denote by I^* some anti-intuitionistic logic of our hierarchy *AC* (excluding Johannsson's dual J^*) and by I the correspondent intuitionistic counterpart, then

(a) $\vdash_I \neg\psi \Leftrightarrow \vdash_{I^*} \psi^*$.
(b) $\vdash_I \psi \Rightarrow \vdash_{I^*} \neg^*\psi^*$.
(c) $\vdash_{I^*} \neg^*\psi^* \not\Rightarrow \vdash_I \psi$.

The proof of the last remark is quite straightforward. A counterexample for item (c) for the logics in the hierarchy *AC* without Johansson's dual is obtained by considering that the law of the excluded middle is not valid in the intuitionistic logics, while the law of non-contradiction is always validated in our hierarchy. These results show that the method of dualization logics preserve a lot of properties for intuitionistic logics.

Now we will see that our obtained hierarchy, although formed by paraconsistent logics, does not coincide with the best known paraconsistent calculi.

First of all, it is clear, that J^* does not coincide with any term of da Costa's hierarchy $(C_n)_{n\in\omega}$ and C_ω (of [12]), because the law of *tertium non datur* is not valid in J^*. For the same reasons, J^* does not coincide either with the calculi $C_{\min}$, $C_{\lim}$, or with $C_n^{\neg\neg}$ (for these see [9]).

For Heyting's dual H^*, we can easily see that H^* does not coincide with any of the C_n, because $\neg\varphi, \varphi, \varphi^\circ \nVdash_{H^*} \psi$, where $\varphi^\circ := \neg(\varphi \wedge \neg\varphi)$. For the same reasons, the anti-intuitionistic logics G_n^* do not coincide with any element of da Costa's hierarchy in [12]. Because our anti-intuitionistic logics do not satisfy the equivalence between a formula and its double negation, none of our logics coincides with *Pac* (cf. [3]), although G_3^* looks very similar to *Pac*. As *LFI*1 is a conservative extension of *Pac* it clearly cannot coincide with any of our anti-intuitionistic logics.

In [9] (Fact 3.64) it was remarked that the De Morgan's Law $\neg(\varphi \wedge \psi) \to \neg\varphi \vee \neg\psi$ is not provable in C_ω nor in **Ci**. But an easy calculation—via semantics—shows that:

$$\neg(\varphi \wedge \psi) \vdash_{H^*} \neg\varphi \vee \neg\psi \quad \text{and} \quad \neg(\varphi \wedge \psi) \vdash_{G_n^*} \neg\varphi \vee \neg\psi.$$

Therefore, neither H^* nor any logic in the hierarchy $(G_n^*)_{n\in\omega}$ coincide with any of C_ω, **Ci**, $C_{\min}$ and **bC**.

On the other hand, it is shown in [9] (p. 65) that the sentence:

$$(\varphi \wedge \neg\varphi) \to \neg\neg(\varphi \wedge \neg\varphi),$$

is valid in each of the logics $(C_n)_{n\in\omega}$ of [12], while it is easily seen to fail in H^* and in G_n^*. Thus, the whole hierarchy $(C_n)_{n\in\omega}$, plus C_ω, is disjoint from our anti-constructive logics.

6.2. The anti-paracomplete logics AP

Although the hierarchy *AC* is a genuinely new family of paraconsistent logics, the second anti-intuitionistic (or, anti-paracomplete) hierarchy *AP* obtained by dualization in Section 5

$$I^{1*} \supsetneq I^{2*} \supsetneq \cdots \supsetneq I^{n*} \supsetneq \cdots \qquad (AP)$$

coincides with the paraconsistent hierarchy $(P^n)_{n\in\omega}$.

The hierarchy *AP* has several other properties which differentiate it from the hierarchy *AC*: not all classical tautologies are valid, and here it is possible, contrary to *AC*, to introduce an internal implication symbol. Proposition 5.2 gives also another relationship between I^n and the dual I^{n*}, as remarked in 6.1 for the first hierarchy.

Another relevant point is that the method of dualizing permits to establish systems of counter-axioms for the hierarchies $(I^n)_{n\in\omega}$ and $(P^n)_{n\in\omega}$, having Dual Modus Ponens as a counter-inference rule. Therefore, we have, for the two hierarchies $(I^n)_{n\in\omega}$ and $(P^n)_{n\in\omega}$, on the one side an axiomatics and on the other side a counter-axiomatics as well.

7. Summary and conclusions

The duality between intuitionistic and paraconsistent logic has remained a thought-provoking question for quite a long time, but with the exception of isolated systems, was never treated methodologically. Dualization was known until now only for Heyting's calculus, treated in [19,24,30,31,37]. In this paper we have defined a general notion of duality and a dualization procedure which applies to a wide class of logics, generalizing previous approaches. This notion of dualization was applied for dualizing two hierarchies of intuitionistic logics. For these two hierarchies we obtained different results, summarized as follows:

- The first hierarchy *AC* seems not to coincide with any known paraconsistent calculus, and (excluding Johansson's dual) validates all classical tautologies.
- The second hierarchy *AP* coincides with the paraconsistent hierarchy $(P^n)_{n\in\omega}$ of [16] and does not validate all classical tautologies.
- There are intuitionistic and paraconsistent calculi which can be shown to coincide with their duals (in the sense of Corollary 5.5 and only after introducing internal implication symbols) as the calculi $I^nP^n,\ n \in \omega$. However, there remains the question which kind of logics are self-dual and how can we characterize them.

Having established the first steps on a useful and widely applicable mathematical notion of duality, our investigation leaves still untouched the deeper philosophical meaning of dualization, as for example: Is there any intrinsic meaning associated with the self-dual logics? Are there alternative senses of dualization, appropriate for other logics? From the more technical side, all the above defined anti-intuitionistic logics seem to be amenable to algebrization using the well-known Blok–Pigozzi methods; will they give raise to interesting algebraic contents? All these questions are left with the hope of stimulating further research.

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